Comment. Math.
Univ. St. Pauli
XX-2, 1971

Periodic solutions of the equation $\ddot{x}+f(x)\dot{x}+g(x)=0$ without the hypothesis $xg(x)>0$ for $|x|>0$

by

Yûkiti SATÔ

(Received April 8, 1971)

1. Introduction

In this paper we are concerned with the equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \qquad \left(\cdot=\frac{d}{dt}\right) \tag{1.1}$$

without the hypothesis $xg(x)>0$ for $|x|>0$. The case where $xg(x)>0$ for $|x|>\tilde{\alpha}>0$ and $xg(x)<0$ for $\tilde{\alpha}>|x|>0$ was investigated by S. Furuya [1] and the author [2]. The case where $g(x)<0$ for $-\tilde{\alpha}<x<0$ and $g(x)>0$ for $x<-\tilde{\alpha}$ or $0<x$ was discussed by Sansone-Conti [3]. Our aim is to obtain more precise and concrete criteria than those of Sansone-Conti concerning the existence and uniqueness of periodic solutions of (1.1).

Suppose throughout this paper that the functions $f(x)$ and $g(x)$ are continuous for all x and satisfy the following conditions:

(i) *there exist two numbers* $\beta_2<\beta_1$ *such that*

$$f(x)<0 \text{ for } \beta_2<x<\beta_1 ,$$
$$f(x)>0 \text{ for } x<\beta_2 \text{ or } x>\beta_1 ;$$

(ii) *there exists a constant* $\alpha<0$ *such that*

$$g(x)>0 \text{ for } x<\alpha \text{ or } x>0 ,$$
$$g(x)<0 \text{ for } \alpha<x<0 ;$$

(iii) $g(x)$ *is Lipschitzian in every finite interval;*

(iv) $g(x)$ *is differentiable at* $x=\alpha$ *and* $x=0$, *and* $g'(\alpha)<0$ *and* $g'(0)\geqq 0$.

If we set

$$F(x)=\int_0^x f(s)ds ,$$

then the transformation $y=\dot{x}+F(x)$ takes the equation (1.1) into the system

$$\dot{x} = y - F(x) \tag{1.2}$$
$$\dot{y} = -g(x) \ .$$

Since $g(x)$ vanishes only at $x=0$ and $x=\alpha$, the right hand sides of (1.2) vanishes simultaneously only at the two points $A_0 = (\alpha, F(\alpha))$ and $O=(0, 0)$, which are the only (finite) singular points of (1.2).

We shall suppose that: $\beta_i \neq 0$ $(i=1, 2)$. Then the origin O is an elementary singular point which is an attractor for $t \to +\infty$ or $t \to -\infty$ according as $f(0)>0$ or $f(0)<0$. Hence, O is stable if $f(0)>0$ and unstable if $f(0)<0$. The index of the point O is $+1$.

Next we examine the nature of the singular point $A_0 = (\alpha, F(\alpha))$. From the definition of $F(x)$ and the hypothesis (iv), we have

$$F(\alpha+X) = F(\alpha) + f(\alpha)X + o(X) \ ,$$
$$g(\alpha+X) = g'(\alpha)X + o(X)$$

for $|X|$ sufficiently small. By the translation $x = X+\alpha$, $y = Y+F(\alpha)$, (1.2) is transformed into the system

$$\dot{X} = -f(\alpha)X + Y + o(X) \ ,$$
$$\dot{Y} = -g'(\alpha)X + o(X) .$$

It follows from the hypothesis (ii), $g'(\alpha)<0$, that the point A_0 is a saddle point for (1.2) and consequently that the index of the point A_0 is -1.

Equation (1.1) has clearly trivial solutions $x(t)=0$ and $x(t)=\alpha$. The possibility of the existence of non-constant periodic solutions will be discussed in § 3.

We note that since the index of O is $+1$ and that of A_0 is -1, a closed path of (1.2) must contain in its interior only a singular point O.

2. Non-existence of periodic solutions

In this section we shall give several sufficient conditions for (1.1) having no non-constant periodic solutions.

We have the following

THEOREM 1. *Suppose that*

(a) $\beta_2 < 0 < \beta_1$;

(b) $F(\alpha) \geqq 0$;

(c) *either* $\int_0^\alpha g(s)\,ds \leqq \int_0^{\delta_1} g(s)\,ds$ *or* $F(x_1) > F(x_2)$ *for any values* x_1 *and* x_2 *of* x *such that* $\alpha \leqq x_1 < 0$, $x_2 > \delta_1$ *and* $\int_0^{x_1} g(s)\,ds = \int_0^{x_2} g(s)\,ds$, *where* $F(\delta_1)=0$, $\delta_1 > 0$.

Then (1.1) *has no periodic solutions other than the constant solutions* $x=0$, $x=\alpha$.

Proof. It is obvious that there exists no closed paths which are contained in the half plane $x<\alpha$ and no closed paths which are contained in the half plane $x>0$. In fact, neither the half plane $x<\alpha$ nor $x>0$ contains singular points.

We shall prove that

(A) *the strip domain* $\alpha\leqq x\leqq\delta_1$ *cannot completely contain a closed path.*

Suppose the contrary, that is, that there exists a closed path Γ with period T which is contained in the strip $\alpha\leqq x\leqq\delta_1$.

Putting

$$G(x)=\int_0^x g(s)\,ds\ ,$$

consider the function

$$\lambda(x,y)=\frac{1}{2}y^2+G(x)\ . \tag{2.1}$$

Along any trajectory $(x(t), y(t))$ of (1.2), we have

$$\frac{d\lambda}{dt}=-g(x)F(x)\ . \tag{2.2}$$

Integrating (2.2) along Γ from $t=0$ to $t=T$, we get by virtue of the assumptions (a) and (b),

$$0=\lambda(A)-\lambda(A)=-\int_0^T g(s)F(s)ds>0\ .$$

where A is the point of Γ corresponding to both $t=0$ and T, which is a contradiction. This proves the assertion (A).

Suppose that there exists a closed path Γ of (1.2). From the preceding, Theorem 1 will be completely proved if we shall show that Γ can cross neither the line $x=\alpha$ nor the line $x=\delta_1$.

First we shall prove that Γ can not traverse the line $x=\alpha$. Contrarily suppose that Γ intersects the line $x=\alpha$. It is clear that there are at least two intersections of Γ and $x=\alpha$. Let $A=(\alpha, y_1)$ and $B=(\alpha, y_2)$ be two adjacent points on Γ at which Γ passes across the line $x=\alpha$ for $t=t_1$ and t_2. Then we have $\dot{x}(t_1)\cdot\dot{x}(t_2)<0$.

On the other hand, we see from the first equation of (1.2) that

$$\dot{x}(t)>0 \text{ for } x(t)=\alpha,\ y(t)>F(\alpha)$$

and

$$\dot{x}(t)<0 \text{ for } x(t)=\alpha,\ y(t)<F(\alpha)\ ,$$

which shows either $y_1 > F(\alpha) > y_2$ or $y_1 < F(\alpha) < y_2$. This fact implies that there are precisely two intersections of Γ and $x=\alpha$ and that Γ encloses the singular point A_0. The second assertion is absurd. Thus Γ can not transverse the line $x=\alpha$.

Next we shall derive a contradiction supposing that Γ cuts the line $x=\delta_1$. By the same argument as above we conclude that Γ cuts the line $x=\delta_1$ in precisely two points, which we denote by $A=(\delta_1, \eta_1)$, $B=(\delta_1, \eta_2)$. We may suppose that $\eta_1>0$ and $\eta_2<0$ because of $F(\delta_1)=0$.

The closed path Γ is divided into two arcs Γ_1, Γ_2 with end points A, B, the former being contained in $x \leqq \delta_1$ and the latter in $x \geqq \delta_1$. It is easily seen that the variable point $(x(t), y(t))$ moves on Γ_1 from B to A and on Γ_2 from A to B as t increases.

Since $d\lambda/dt$ is nonnegative for $x \leqq \delta_1$ and nonpositive for $x \geqq \delta_1$, we have the inequalities

$$\lambda(A) \geqq \lambda(x(t), y(t)) \geqq \lambda(B) \tag{2.3}$$

for any point $(x(t), y(t))$ on Γ.

The closed path Γ is mapped into a curve in the (λ, y)-plane by (2.1). We denote by $y=y_{\mathrm{I}}(\lambda)$ and $y=y_{\mathrm{II}}(\lambda)$ $(b \leqq \lambda \leqq a)$ the images of Γ_1 and Γ_2 respectively, where $a=\lambda(A)$ and $b=\lambda(B)$. Since the function $\lambda(x(t), y(t))$ is monotone on Γ_1 and Γ_2, the functions $y_{\mathrm{I}}(\lambda)$ and $y_{\mathrm{II}}(\lambda)$ are single-valued functions in λ.

From (1.2), (2.1), and (2.3), it follows that $y_{\mathrm{I}}(\lambda)$ and $y_{\mathrm{II}}(\lambda)$ satisfy the equation

$$\frac{dy(\lambda)}{d\lambda} = \frac{1}{F(x)} \quad \text{for} \quad \lambda(A) > \lambda > \lambda(B)\ , \tag{2.4}$$

from which

$$\lim_{\lambda \to a} y_{\mathrm{I}}'(\lambda) = -\infty\ , \qquad \lim_{\lambda \to a} y_{\mathrm{II}}'(\lambda) = +\infty\ ,$$
$$\lim_{\lambda \to b} y_{\mathrm{I}}'(\lambda) = -\infty\ , \qquad \lim_{\lambda \to b} y_{\mathrm{II}}'(\lambda) = +\infty \qquad ('=d/d\lambda)\ .$$

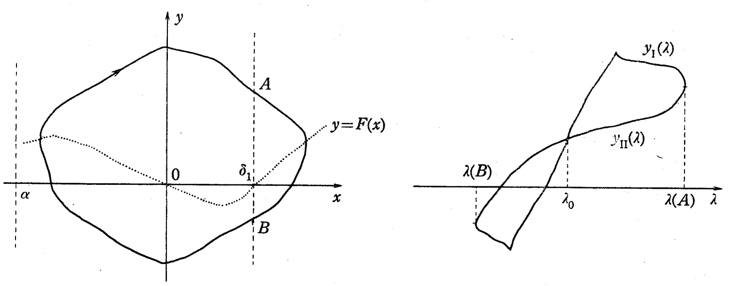

Fig. 1.

Hence, for λ sufficiently close to a (or b), we have $y_{\mathrm{I}}(\lambda) > y_{\mathrm{II}}(\lambda)$ (or $y_{\mathrm{I}}(\lambda) < y_{\mathrm{II}}(\lambda)$). Therefore there exists a λ_0, $b < \lambda_0 < a$, such that $y_{\mathrm{I}}(\lambda_0) =$

$y_{\mathrm{II}}(\lambda_0)$ (see Fig. 1). Let (x_1, y_1) and (x_2, y_2) be the points on Γ corresponding to $(\lambda_0, y_{\mathrm{I}}(\lambda_0))$ and $(\lambda_0, y_{\mathrm{II}}(\lambda_0))$ respectively. Then we have

$$\lambda_0=\frac{1}{2}y_1^2+G(x_1)=\frac{1}{2}y_2^2+G(x_2) ,$$

and hence

$$G(x_1)=G(x_2) .$$

By the hypothesis (ii), $G(x)$ is decreasing for $\alpha<x<0$ and increasing for $0<x$. Thus we have

$$G(\alpha)>G(x_1)=G(x_2)>G(\delta_1) .$$

The inequality $G(\alpha)>G(\delta_1)$ contradicts the first condition of assumption (c).

Next we shall prove that the equality $G(x_1)=G(x_2)$ contradicts also the second condition of (c). If we put $\mu=\max\{\lambda; y_{\mathrm{I}}(\lambda)=y_{\mathrm{II}}(\lambda), a>\lambda>b\}$, then we have by (2.4)

$$\left.\frac{dy_{\mathrm{I}}(\lambda)}{d\lambda}\right|_{\lambda=\mu}=\frac{1}{F(x_1)}\geqq\frac{1}{F(x_2)}=\left.\frac{dy_{\mathrm{II}}(\lambda)}{dy}\right|_{\lambda=\mu}$$

and hence

$$F(x_1)\leqq F(x_2) , \qquad G(x_1)=G(x_2) ,$$

which contradicts (c).

The proof of Theorem 1 is thus completed.

The assumption (b), $F(\alpha)\geqq 0$, in Theorem 1 can be replaced by $F(\alpha)<0$ if both of the alternative conditions in (c) are supposed to hold. That is, we have the

THEOREM 2. *With the above notation, suppose that*

(a) $\beta_2<0<\beta_1$;

(b) $F(\alpha)<0$;

(c) $G(\alpha)\leqq G(\delta_1)$;

(d) *for* $x_1>x_2>\alpha$ *such that* $G(x_1)=G(x_2)$, *we have* $F(x_1)<F(x_2)$.

Then there is no periodic solutions of (1.1).

Proof. Suppose that there exists a closed path Γ: $x=x(t)$, $y=y(t)$ of (1.2). By discussions analogous to the preceding, we can prove that the closed path Γ is contained neither in the half plane $\delta_2\leqq x$ nor in $x\leqq\delta_2$, where $F(\delta_2)=0$, $\delta_2<0$.

Now suppose that Γ is contained in the strip $\alpha\leqq x\leqq\delta_1$. We can consider, as above, y as a function of λ and this function is decomposed into two single-valued functions $y_{\mathrm{I}}(\lambda)$ and $y_{\mathrm{II}}(\lambda)$ which correspond to the parts of Γ lying in $\delta_2\leqq x\leqq\delta_1$ and in $\alpha\leqq x\leqq\delta_2$ respectively. Let $A=$

(δ_2, η_1) $(\eta_1>0)$ and $B(\delta_2, \eta_2)$ $(\eta_2<0)$ be the points where Γ cuts the line $x=\delta_2$, and set $\lambda(A)=a$ and $\lambda(B)=b$. Then we have $a<b$ and

$$y_{\mathrm{I}}(\lambda)>y_{\mathrm{II}}(\lambda) \quad \text{for } \lambda \text{ near } a\,,$$
$$y_{\mathrm{I}}(\lambda)<y_{\mathrm{II}}(\lambda) \quad \text{for } \lambda \text{ near } b\,.$$

Therefore the curves $y=y_{\mathrm{I}}(\lambda)$ and $y=y_{\mathrm{II}}(\lambda)$ intersect necessarily at some point $\lambda=\lambda_0$. We denote by ν the minimum value of λ of intersection points and let (x_1, y_1) and (x_2, y_2) be the points on the left and right arcs corresponding to this value. We have

$$\alpha<x_2<\delta_2<x_1<\delta_1\,,$$

and moreover

$$G(x_1)=G(x_2)\,,$$

because of the equality

$$\nu=\frac{1}{2}\{y_{\mathrm{I}}(\nu)\}^2+G(x_1)=\frac{1}{2}\{y_{\mathrm{II}}(\nu)\}^2+G(x_2)\,.$$

From $dy_{\mathrm{I}}(\nu)/d\lambda\leqq dy_{\mathrm{II}}(\lambda)/d\lambda$ and (2.4), we have

$$F(x_1)\geqq F(x_2)\,,$$

which contradicts (d).

We can prove in the same way as in the proof of Theorem 1 that the straight line $x=\alpha$ cannot be crossed by Γ. Therefore Γ necessarily cuts both the lines $x=\delta_1$ and $x=\delta_2$. Now let A, B, C, and D be the points where Γ cuts the half lines $\{x=\delta_1, y>0\}$, $\{x=\delta_1, y<0\}$, $\{x=\delta_2, y<0\}$, and $\{x=\delta_2, y>0\}$ respectively. The sketch of the curve Γ is shown in Fig. 2.

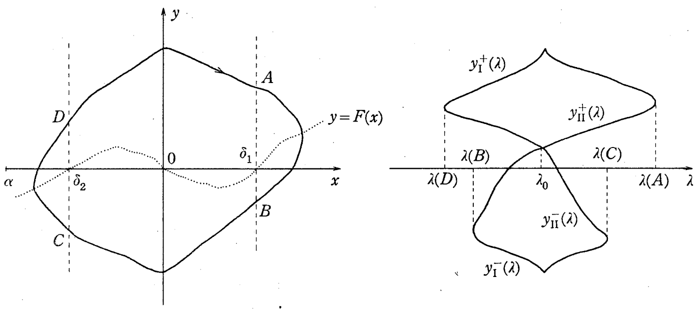

Fig. 2.

We denote by $y_{\mathrm{II}}^{+}(\lambda)$, $y_{\mathrm{I}}^{-}(\lambda)$, $y_{\mathrm{II}}^{-}(\lambda)$, and $y_{\mathrm{I}}^{+}(\lambda)$ the curves on the (λ, y)-plane corresponding to the parts $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, and $\widehat{DA}$ respectively. By discussions analogous to the preceding, we can prove that the curve $y=y_{\mathrm{I}}^{+}(\lambda)$ meets neither the curve $y=y_{\mathrm{II}}^{+}(\lambda)$ nor the curve

$y=y_{\mathrm{II}}^{-}(\lambda)$ except for $\lambda=\lambda(A)$, $\lambda(D)$ and that the curve $y=y_{\mathrm{I}}^{-}(\lambda)$ meets neither the curve $y=y_{\mathrm{II}}^{+}(\lambda)$ nor the curve $y=y_{\mathrm{II}}^{-}(\lambda)$ except for $\lambda=\lambda(B)$, $\lambda(C)$. Since $\lambda(A)>\lambda(D)$ and $\lambda(C)>\lambda(B)$, the curve $y=y_{\mathrm{II}}^{+}(\lambda)$ must intersect the curve $y=y_{\mathrm{II}}^{-}(\lambda)$ at some value $\lambda=\lambda_0$ (see Fig. 2).

This implies that

$$\lambda_0=\frac{1}{2}\{y_{\mathrm{II}}^{+}(\lambda_0)\}^2+G(x^+)=\frac{1}{2}\{y_{\mathrm{II}}^{-}(\lambda_0)\}^2+G(x^-)\ ,$$

$$x^+>\delta_1,\ x^->\delta_2\ ,$$

where x^+ and x^- denote the abscissas of the points on Γ corresponding to the value λ_0. Therefore

$$G(\delta_1)<G(x^+)=G(x^-)<G(\alpha)\ ,$$

which is impossible by assumption (c).

The theorem is thus established.

Next we shall prove the following

THEOREM 3. *Suppose that*

(a) $\beta_2<\alpha,\ 0<\beta_1$;
(b) *either* $G(\beta_1)\geqq G(\alpha)$, *or*

$$\frac{g(x_1)}{f(x_1)}<\frac{g(x_2)}{f(x_2)} \tag{2.5}$$

for x_1 *and* x_2 *such that* $\alpha<x_1<0$, $\beta_1<x$, *and* $G(x_1)=G(x_2)$. *Then* (1.1) *does not have non-constant periodic solutions.*

Proof. Consider instead of the equation (1.1) the equivalent system

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-f(x)y-g(x)\ .\end{aligned} \tag{2.6}$$

We shall need the examination of the variation of the function

$$\lambda(x,\ y)=\frac{1}{2}y^2+G(x) \tag{2.7}$$

along any trajectory of (2.6). For this purpose we use

$$\frac{d\lambda}{dt}=-y^2f(x)\ , \tag{2.8}$$

and

$$\frac{dy}{d\lambda}=\frac{1}{y}+\frac{g(x)}{y^2f(x)}\ . \tag{2.9}$$

Let x_0 be an arbitrary value such that $0>x_0\geqq\alpha$ and consider a trajectory Γ of (2.6) passing through the point $A=(x_0,\ 0)$. Let B and

C (resp. D and E) (in Fig. 3) be the points of Γ at which Γ intersects for the first time the lines $x=0$ and $y=0$, as t increases (resp. decreases) starting from the point A, if there exist such points.

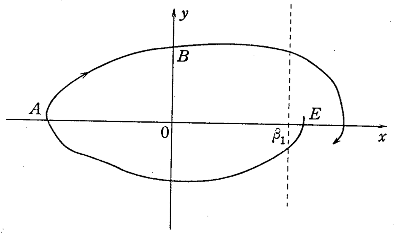

Fig. 3.

It is easily seen that any closed path of (2.6) crosses necessarily the x-axis at a point P with abscissa $\tilde{x}$, $\alpha<\tilde{x}<0$. Thus, to complete our proof, we have only to show the following assertion.

(A) *C is to the right of E.*

Since $G(x)$ is increasing for $0<x$, (A) is equivalent to

$$\lambda(E)<\lambda(C) , \tag{2.10}$$

which we shall prove.

Consider the case where the arc $\widehat{AC}$ does not cross the line $x=\beta_1$. Then λ is an increasing function of t along $\widehat{AC}$ and consequently

$$\lambda(A)<\lambda(C) . \tag{2.11}$$

Next we consider the case where the arc $\widehat{AC}$ intersects the line $x=\beta_1$ at $B'=(\beta_1, y_0)$. We denote by $y=y_{\text{I}}(\lambda)$ the curve in the (λ, y)-plane which corresponds to the arc $\widehat{AB'}$ and by $y=y_{\text{II}}(\lambda)$ the curve which corresponds to the arc $\widehat{B'C}$. We have, by virtue of (2.8), (2.9),

$$\lim_{\lambda\to b'} y'_{\text{I}}(\lambda)=-\infty , \quad \lim_{\lambda\to b'} y'_{\text{II}}(\lambda)=+\infty , \quad \left('=\frac{d}{d\lambda},\ b'=\lambda(B')\right) .$$

Hence, when λ $(<b')$ is sufficiently close to b', we get

$$\frac{dy_{\text{I}}(\lambda)}{d\lambda}<\frac{dy_{\text{II}}(\lambda)}{d\lambda}$$

and so

$$y_{\text{I}}(\lambda)>y_{\text{II}}(\lambda) .$$

Suppose that there exists a λ $(<b')$ such that $y_{\text{I}}(\lambda)=y_{\text{II}}(\lambda)$. Setting

$$\lambda_0=\max\{\lambda;\, y_{\text{I}}(\lambda)=y_{\text{II}}(\lambda),\ \lambda<b'\} ,$$

we have

$$y'_{\mathrm{I}}(\lambda_0) \geqq y'_{\mathrm{II}}(\lambda_0) \; . \tag{2.12}$$

Let $\lambda_0=\lambda(x_1, y_{\mathrm{I}}(\lambda_0))=\lambda(x_2, y_{\mathrm{II}}(\lambda_0))$, then we have

$$G(x_1)=G(x_2),\; x_1<0,\; \beta_1<x_2 \; .$$

This contradicts $G(\beta_1)\geqq G(\alpha)$. Therefore we have $y_{\mathrm{I}}(\lambda)>y_{\mathrm{II}}(\lambda)$. From this we conclude that (2.11) holds under the assumption that $G(\beta_1)\geqq G(\alpha)$.

Then we consider the case where $G(\beta_1)<G(\alpha)$. If $y_{\mathrm{I}}(\lambda_0)\neq 0$, then (2.9) and (2.12) imply the inequality

$$\frac{g(x_1)}{f(x_1)} \geqq \frac{g(x_2)}{f(x_2)} \; ,$$

which contradicts (2.5). If $y_{\mathrm{I}}(\lambda_0)=0$, then $\lambda_0=a=c$ and $y_{\mathrm{I}}(\lambda)>y_{\mathrm{II}}(\lambda)>0$ for $a<\lambda<b'$. Therefore for λ sufficiently close to a, we have $y'_{\mathrm{I}}(\lambda)<y'_{\mathrm{II}}(\lambda)$. This implies that there exists a λ_1 such that $y_{\mathrm{I}}(\lambda_1)=y_{\mathrm{II}}(\lambda_1)$ and $a<\lambda_1<b'$, which gives also a contradiction. Thus in the case that $G(\beta_1)<G(\alpha)$, (2.11) holds.

In the same way we can prove

$$\lambda(E)<\lambda(A) \; .$$

This completes the proof of (2.10).

In the case that $\beta_1<0$, we have the following theorems.

THEOREM 4. *Suppose that the following conditions are satisfied:*
(a) $\beta_2\leqq\alpha<\beta_1<0$;
(b) *for* $\tilde{x}$, $\underset{\sim}{x}$, $\alpha\leqq\tilde{x}<\beta_1$, $0<\underset{\sim}{x}$ *such that* $G(\tilde{x})=G(\underset{\sim}{x})$, *it holds that*

$$\tilde{x}+\underset{\sim}{x}>2\beta_1 \; ;$$

(c) *for* $\alpha<x_1<\beta_1<x_2<\gamma$, *the inequality*

$$f(x_1)+f(x_2)\geqq 2f\left(\frac{x_1+x_2}{2}\right)$$

holds, where γ *is a positive number such that* $G(\gamma)=G(\alpha)$. *Then* (1.1) *cannot have non-constant periodic solutions.*

THEOREM 5. *Suppose that*
(a) $\beta_2\leqq\alpha<\beta_1<0$;
(b) *for* x_1, x_2, $\alpha<x_1<\beta_1$, $0<x_2$ *such that* $G(x_1)=G(x_2)$, *we have*

$$F(x_1)<F(x_2) \; .$$

Then there is no periodic solutions of (1.1).

THEOREM 6. *If* $\beta_1\leqq\alpha$, *then* (1.1) *has no periodic solutions.*

Theorem 5 is proved without any essential modification of the reasonings used in the proof of Theorem 2. The proof of Theorem 6 is easy. Therefore we shall give only the proof of Theorem 4.

Proof of Theorem 4. We consider the equivalent system (2.6), and examine the behavior of a trajectory Γ of (2.6) passing through a point $A=(x_0, 0)$, where $\alpha \leqq x_0 < \beta_1$.

By the same reasoning as in the proof of Theorem 3, we can limit our attention to the case sketched in Fig. 4, i.e. the case where there exsit successive intersection points E, C of Γ and the x-axis.

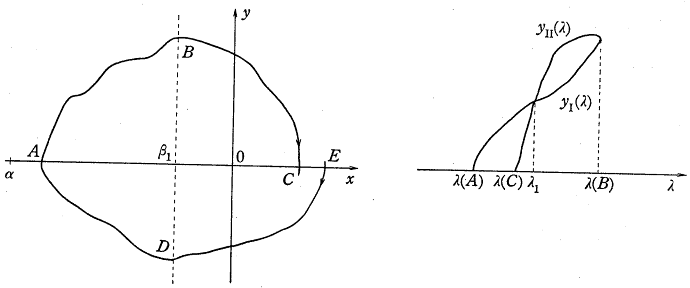

Fig. 4.

We shall show that for any x_0 such that $\alpha \leqq x_0 < \beta_1$ we have

$$\lambda(C) < \lambda(A) , \tag{2.13}$$

where, as above, $\lambda(A)$ and $\lambda(C)$ denote the values of $\lambda(x, y)=(1/2)y^2+G(x)$ at A, C (in Fig. 4).

Keeping x_0 fixed, we suppose the contrary, that is, that

$$\lambda(C) \geqq \lambda(A) .$$

Since λ increases along the arc of the trajectory Γ for $\alpha < x < \beta_1$, when t increases, it follows that the value b of λ at B on Γ (see Fig. 4) is greater than the value a of λ at A.

We denote by $y_{\text{I}}(\lambda)$ the function of λ which corresponds to the arc $\widehat{AB}$ of Γ, and by $y_{\text{II}}(\lambda)$ which corresponds to the arc $\widehat{BC}$ of Γ. From (2.9) we have

$$\lim_{\lambda \to b} y'_{\text{I}}(\lambda) = +\infty , \qquad \lim_{\lambda \to b} y'_{\text{II}}(\lambda) = -\infty ,$$

and hence, for λ sufficiently close to b, we have

$$y'_{\text{I}}(\lambda) > y'_{\text{II}}(\lambda) ,$$

which yields

$$y_{\text{I}}(\lambda) < y_{\text{II}}(\lambda) .$$

Therefore there exists a value of λ (see Fig. 4) such that

$$c \leqq \lambda < b \quad \text{and} \quad y_{\text{I}}(\lambda) = y_{\text{II}}(\lambda) .$$

Let $\lambda_1=\max\{\lambda;\, y_{\rm I}(\lambda)=y_{\rm II}(\lambda),\ \lambda<b\}$, and let (x_1, y_1), (x_2, y_2) be the coordinates of the points on the curve Γ which correspond to the value λ_1. By definition, we have

$$x_1<\beta_1<x_2 .$$

Moreover we have

$$G(x_1)=G(x_2)\ ,\quad x_0\leqq x_1<\beta_1\ ,\quad 0<x_2\ ,$$

from which and from the conditions (b) and (c), we have

$$f(x_1)+f(x_2)\geqq 2f\left(\frac{x_1+x_2}{2}\right)>0 .$$

Hence we get

$$-f(x_1)<f(x_2)\ ,\quad x_1+x_2>2\beta_1 . \tag{2.14}$$

We consider x as a function of λ. Then this function is in a natural way decomposed into two single-valued functions $x_{\rm I}(\lambda)$ and $x_{\rm II}(\lambda)$ which correspond to the parts of Γ lying in $x_1\leqq x\leqq\beta_1$ and $\beta_1\leqq x\leqq x_2$ respectively.

Now it follows from the definition of λ_1 that

$$y_{\rm II}(\lambda)>y_{\rm I}(\lambda)>0 \quad \text{for} \quad \lambda_1<\lambda<b ,$$

and from $-f(x_1)<f(x_2)$ that

$$-f(x_{\rm I}(\lambda))<f(x_{\rm II}(\lambda)) \quad \text{for} \quad \lambda\ (>\lambda_1) \text{ near } \lambda_1 .$$

Therefore we have, for λ $(>\lambda_1)$ near λ_1,

$$\begin{aligned}\frac{dx_{\rm I}(\lambda)}{d\lambda}&=-\frac{1}{y_{\rm I}(\lambda)f(x_{\rm I}(\lambda))}\\ &>\frac{1}{y_{\rm II}(\lambda)f(x_{\rm II}(\lambda))}=-\frac{dx_{\rm II}(\lambda)}{d\lambda} .\end{aligned}$$

Since $x_1+x_2>2\beta_1$, the integration of both sides of this inequality from λ_1 to b yields

$$\int_{\lambda_1}^{b}\frac{dx_{\rm I}(\lambda)}{d\lambda}d\lambda=\beta_1-x_1<x_2-\beta_1=\int_{\lambda_1}^{b}-\frac{dx_{\rm II}(\lambda)}{d\lambda}d\lambda ,$$

so that there exists a λ_2 such that

$$\frac{dx_{\rm I}(\lambda_2)}{d\lambda}=\frac{dx_{\rm II}(\lambda_2)}{d\lambda} \quad\text{and}\quad \frac{dx_{\rm I}(\lambda)}{d\lambda}>-\frac{dx_{\rm II}(\lambda)}{d\lambda} \tag{2.15}$$

for $\lambda_1<\lambda<\lambda_2$. As $0<y_{\rm I}(\lambda_2)<y_{\rm II}(\lambda_2)$, we have

$$-f(x_{\rm I}(\lambda_2))>f(x_{\rm II}(\lambda_2)) . \tag{2.16}$$

On the other hand, integrating both sides of (2.15) from λ_1 to λ_2, we obtain

$$x_{\mathrm{I}}(\lambda_2)-x_1=\int_{\lambda_1}^{\lambda_2}\frac{dx_{\mathrm{I}}(\lambda)}{d\lambda}d\lambda$$
$$>\int_{\lambda_1}^{\lambda_2}-\frac{dx_{\mathrm{II}}(\lambda)}{d\lambda}d\lambda=x_2-x_{\mathrm{II}}(\lambda_2)\ ,$$

which implies by (2.14)

$$x_{\mathrm{I}}(\lambda_2)+x_{\mathrm{II}}(\lambda_2)>2\beta_1\ .$$

Therefore by virtue of the assumption (c),

$$-f(x_{\mathrm{I}}(\lambda_2))<f(x_{\mathrm{II}}(\lambda_2))\ ,$$

which contradicts (2.16). Thus (2.13) holds.

If we now proceed from E to A in much the same way as from A to C, it follows that the value of λ at E is greater than $\lambda(A)$.

Therefore we have

$$\lambda(C)<\lambda(E)\ ,$$

which implies that there exists no closed path of (2.6).

This completes the proof of Theorem 4.

Finally we consider the case that $0<\beta_2$.

THEOREM 7. *Suppose that*

(a) $0<\beta_2$;

(b) *either* $F(x)\geqq 0$ *for* $x\geqq 0$ *or* $F(x_1)>F(x_2)$ *for* $\alpha<x_2<0$, $\delta_2<x_1$ *such that* $G(x_1)=G(x_2)$, *where* $\delta_2=\min\{x;\ F(x)=0,\ 0<x\}$.

Then (1.1) *cannot have periodic solutions.*

THEOREM 8. *Suppose that*

(a) $0<\beta_2$;

(b) $G(\alpha)\leqq G(\beta_1)$;

(c) *either* $G(\alpha)\leqq G(\beta_2)$ *or* $g(x_1)/f(x_1)>g(x_2)/f(x_2)$ *for* $\alpha<x_2<0$, $\beta_2<x_1<\beta_1$ *such that* $G(x_1)=G(x_2)$.

Then there is no periodic solutions of (1.1).

Since the proofs of Theorems 7 and 8 are essentially similar to those of Theorems 2 and 3 respectively, we omit the proofs.

3. Existence of periodic solutions

For simplicity's sake, we use the following notations.

$$(3.1)\qquad \begin{aligned} &G=\max\{G(\beta_1),\ G(\beta_2)\}\ , && C=|G(\alpha)-G(\beta_2)|^{1/2}\ .\\ &M=G/C-F(\beta_1)+F(\beta_2)\ , && y_0=\sqrt{2}\,C\ ,\\ &\lambda_0=G(\alpha)\ , && \theta=y_0/M\ . \end{aligned}$$

We have the following

THEOREM 9. *Suppose that*

(a) $\alpha<\beta_2<0<\beta_1$;

(b) *there exist two numbers* x_1, $\alpha<x_1<\beta_2$, *and* $\omega>1$ *such that*

$$2\omega(1+1/\theta)M\leqq\int_{x_1}^{\beta_2}f(x)dx\ , \tag{3.2}$$

and

$$G(\alpha)-G(\beta_2)\leqq\omega^2(G(\alpha)-G(x_1))\ ; \tag{3.3}$$

(c) $$\lim_{n\to\infty}G(x)>M+y_0\ .$$

Then (1.1) *has at least one non-constant periodic solution.*

Proof. Consider the equivalent system

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-f(x)y-g(x)\ .\end{aligned} \tag{3.4}$$

This system admits two singular points: the saddle point $A_0=(\alpha, 0)$ and the unstable point $O=(0, 0)$, and admits four separatrices Γ_+, Γ_-, Π_+, and Π_- which are depicted in Fig. 5 by a heavy curve.

Consider the separatrix Γ_+. It leaves A_0 for t increasing and enters a region $\Omega=\{\beta_2>x>\alpha,\ -g(x)/f(x)>y>0\}$, where $\dot{x}>0$, $\dot{y}>0$.

We shall prove that Γ_+ is a contracting spiral. There are several cases to be considered.

First suppose that the separatrix Γ_+ intersects the line $x=\beta_1$ and let $P_1=(\beta_2, y_1)$, $P_2=(0, y_2)$, and $P_3=(\beta_1, y_3)$ denote the successive intersections of Γ_+ and the lines $x=\beta_2$, $x=0$, and $x=\beta_1$, for t increasing, which are in the order as indicated in Fig. 5.

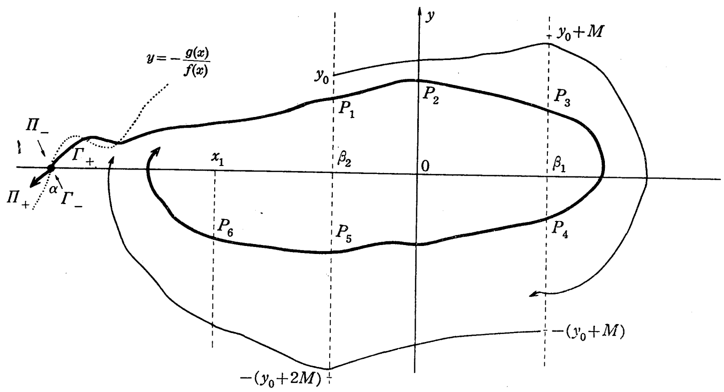

Fig. 5.

Since $\lambda(x, y)\,(\equiv(1/2)y^2+G(x))$ is decreasing along an arbitrary integral curve if $x\leqq\beta_2$, it follows that $\lambda(P_1)<\lambda_0$ and hence that $y_1<y_0$. Now either y_1 is greater than or equal to C, or it is less than C. Let us

suppose that the former is the case. Then all the points on the arc $\widehat{P_1P_2}$ have ordinates greater than or equal to C. In fact, eliminating t in (3.4), we have

$$dy/dx = -f(x) - g(x)/y \;. \tag{3.5}$$

Since $-f(x)-g(x)/y>0$ for $\beta_2 \leqq x \leqq 0$, $y>0$, the solution $\tilde{y}(x)$ of (3.5) such that $\tilde{y}(\beta_2)=C$ is increasing. Thus we have

$$\tilde{y}(x) \geqq C \quad \text{for} \quad \beta_2 \leqq x \leqq 0 \;.$$

We set

$$\varphi(x) = \begin{cases} -f(x)-g(x)/C \;, & \text{for} \quad \beta_2 \leqq x \leqq 0 \;, \\ -f(x) \;, & \text{for} \quad 0 < x \leqq \beta_1 \;. \end{cases}$$

Then we have

$$-f(x)-g(x)/y \leqq \varphi(x) \quad \text{for} \quad y \geqq C \text{ and } \beta_2 \leqq x \leqq \beta_1 \;.$$

We denote by $Y(x)$ the solution of

$$dY/dx = \varphi(x) \;, \tag{3.6}$$

satisfying the initial condition $Y(\beta_2)=y_0$.

Taking account of the hypotheses on $f(x)$ and $g(x)$, and applying a theorem of comparison, we get

$$Y(x) \geqq \tilde{y}(x) \quad \text{for} \quad \beta_2 \leqq x \leqq \beta_1 \;,$$

where $\tilde{y}(x)$ is the solution of (3.5) corresponding to the initial condition $\tilde{y}(\beta_2)=y_1$.

Integrating (3.6) from β_2 to β_1, we have

$$Y(\beta_1) - Y(\beta_2) = -\int_{\beta_2}^{\beta_1} f(x)dx - \int_{\beta_2}^{0} g(x)/C dx \;,$$

hence by (3.1)

$$Y(\beta_1) - y_0 \leqq M \;.$$

Thus

$$y_3 \leqq Y(\beta_1) \leqq y_0 + M \;.$$

From the assumption (c), it follows that the equation in x: $\lambda(x, 0) = M + y_0$ has a positive root, which is denoted by ξ_0.

Since λ is a decreasing function of t along a trajectory of (3.4) if $\beta_1 \leqq x$, the trajectory leaving from P_3 must meet the x-axis between β_1 and ξ_0. Let $P_4 = (\beta_1, y_4)$ (in Fig. 5) be the intersection of this trajectory and the half line $\{x=\beta_1, y<0\}$. Then we have

$$|y_4| \leqq y_3 \leqq y_0 + M \;. \tag{3.7}$$

In the latter case, i.e. the case where $y_1<C$, we consider the trajectory $\tilde{\Gamma}$: $(\xi(t), \eta(t))$ of (3.4) leaving from the point (β_2, C), and denote by t_3 the value of t such that $\xi(t_3)=\beta_1$. Applying theorems of uniqueness and comparison, we have

$$y_3\leqq\eta(t_3)\leqq y_0+M\ ,$$

and consequently (3.7) holds.

In addition to the hypothesis that Γ_+ meets the line $x=\beta_1$, we make an assumption that Γ_+ meets also the half line $\{x=x_1, y<0\}$. Let $P_5=(\beta_2, y_5)$ and $P_6=(x_1, y_6)$ be the first intersection points of Γ_+ and the half lines $\{x=\beta_2, y<0\}$ and $\{x=x_1, y<0\}$ respectively. Proceeding from P_4 to P_5 in the same way as from P_1 to P_3, we have, using (3.7),

$$|y_5|\leqq y_0+2M\ . \tag{3.8}$$

Integrating $d\lambda/dx=-yf(x)$ from P_5 to P_6,

$$\begin{aligned}\lambda(P_6)-\lambda(P_5)&=-\int_{\beta_2}^{x_1}yf(x)dx\\&=-\int_{x_1}^{\beta_2}|y|f(x)dx\ .\end{aligned}$$

If we have always $|y|\geqq(1/\omega)y_0$ along $\widehat{P_5P_6}$, then we have

$$\begin{aligned}\lambda(P_6)-\lambda(P_5)&<-\frac{1}{\omega}y_0\int_{x_1}^{\beta_2}f(x)dx\\&\leqq-2y_0M(1+1/\theta)\ ,\end{aligned}$$

from which, using (3.1) and (3.8),

$$\begin{aligned}\lambda(P_6)&\leqq\frac{1}{2}(y_0+2M)^2+G(\beta_2)-2y_0M(1+1/\theta)\\&=\frac{1}{2}y_0^2+G(\beta_2)+2M(y_0+M)-2y_0M(1+1/\theta)\\&=G(\alpha).\end{aligned}$$

In the case where we have not always $|y|\geqq(1/\omega)y_0$, the above inequality is also derived from the condition that $\dot{y}(t)>0$ for $\alpha<x<\beta_2$ and $y<0$. In fact, from the condition above we deduce $|y_6|\leqq y_0/\omega$, from which and from the assumption (3.3) we have

$$\lambda(P_6)\leqq\frac{1}{2}\left\{\frac{1}{\omega}y_0\right\}^2+G(x_1)=\left\{\frac{C}{\omega}\right\}^2+G(x_1)\leqq G(\alpha)\ .$$

Since $\lambda(x, y)$ decreases, for t increasing, along the part of the curve Γ_+ which lies in the strip $x\leqq\beta_2$, it follows that the separatrix Γ_+ must cut the negative x-axis between x_2 and x_1, where $G(x_2)=\lambda(P_6)$,

$\alpha \leqq x_2 < x_1$. This shows that Γ_+ is a contracting spiral.

If Γ_+ does not meet the half line $\{x=x_1, y<0\}$, then we obtain, by constructing a function similar to $\varphi(x)$, an estimate

$$y \geqq -(y_0+2M) .$$

It follows from this that Γ_+ necessarily cuts the x-axis for $x_1<x<0$. In this case also, Γ_+ is a contracting spiral.

We proceed to the case that Γ_+ does not meet the line $x=\beta_1$. Since Γ_+ is majorized by $Y(x)$ from above, Γ_+ cuts the x-axis in the interval $0<x\leqq\beta_1$. In order to see that Γ_+ is a contracting spiral, it suffices to show that the trajectory starting from the point $(\beta_1, 0)$ necessarily cuts the x-axis in the interval $\alpha<x<0$. This fact follows at once, however, from the proof in the first case.

On the other hand, since the origin is unstable, integral curves starting at points near the origin go away from the origin as t indefinitely increases.

The preceding enables us to construct a region which statisfies the hypotheses of the theorem of Poincaré-Bendixson which assures the existence of at least one closed path.

Thus the theorem is established.

4. A criterion for the uniqueness of periodic solution

In this section we shall give a sufficient condition for the uniqueness of periodic solution.

THEOREM 10. *Suppose that $f(x)$ and $g(x)$ are continuously differentiable. If the conditions* (a) $\alpha<\beta_2<0<\beta_1$ *and* (b) $G(\beta_2)=G(\beta_1)$ *hold, then* (1.1) *admits at most one non-constant periodic solution.*

Proof. We consider instead of (1.1) the equivalent system (3.4). Let Γ be an arbitrary closed path of (3.4).

In the same way as above, we can show that Γ is completely contained in neither the half plane $x\leqq\beta_2$ nor the half plane $\beta_1\leqq x$ nor the strip $\beta_2\leqq x\leqq\beta_1$

We shall first prove that Γ passes across both the lines $x=\beta_1$ and $x=\beta_2$.

Suppose that Γ is contained completely in the strip $\alpha\leqq x\leqq\beta_1$. Γ is divided into two parts Γ_1 and Γ_2, the former lying in $\beta_2\leqq x\leqq\beta_1$, the latter in $\alpha\leqq x\leqq\beta_2$. We denote by $y_{\mathrm{I}}(\lambda)$ and $y_{\mathrm{II}}(\lambda)$ the functions of λ corresponding to Γ_1 and Γ_2 respectively. By discussions analogous to the preceding, we see that there exists a $\tilde{\lambda}$ such that $y_{\mathrm{I}}(\tilde{\lambda})=y_{\mathrm{II}}(\tilde{\lambda})$. Let (x_1, y_1), (x_2, y_2) $(x_i\neq\beta_2)$ be the points on Γ_1, Γ_2 at which λ takes the value $\tilde{\lambda}$. We have then

$$G(x_1)=G(x_2) \quad \text{and} \quad \alpha<x_2<\beta_2<x_1<\beta_1 . \tag{4.1}$$

From the assumption (b), it follows that for any values x_1 and x_2 of x such that $\alpha<x_2<\beta_2<x_1\leqq\beta_1$,

$$G(x_2)>G(\beta_2)\geqq G(x_1) ,$$

which contradicts (4.1). Thus Γ is not completely contained in $\alpha\leqq x\leqq\beta_1$.

In the same way, we can conclude that a closed path Γ is not contained in the half plane $\beta_2\leqq x$.

Thus Γ cuts both the lines $x=\beta_1$ and $x=\beta_2$. Now let A, B, C, D, E, and F be the intersections of Γ with the lines $x=\beta_1$, $x=\beta_2$, and the x-axis as indicated in Fig. 6. We denote by $y_{\mathrm{II}}^{+}(\lambda)$, $y_{\mathrm{I}}^{-}(\lambda)$, $y_{\mathrm{II}}^{-}(\lambda)$,

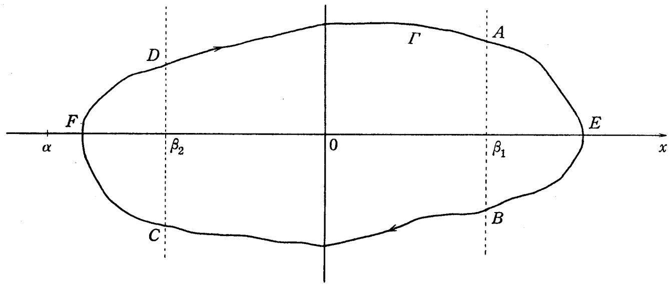

Fig. 6.

and $y_{\mathrm{I}}^{+}(\lambda)$ the functions of λ corresponding to the arcs $\overparen{AB}$, $\overparen{BC}$, $\overparen{CD}$, and $\overparen{DA}$ respectively. Then we can prove, by arguments similar to the preceding, that $y=y_{\mathrm{I}}^{+}(\lambda)$ intersects neither $y=y_{\mathrm{II}}^{+}(\lambda)$ nor $y=y_{\mathrm{II}}^{-}(\lambda)$ except at the points $\lambda=\lambda(A)$ and $\lambda(D)$, and that $y=y_{\mathrm{I}}^{-}(\lambda)$ intersects neither $y=y_{\mathrm{II}}^{+}(\lambda)$ nor $y=y_{\mathrm{II}}^{-}(\lambda)$ except at $\lambda=\lambda(B)$, $\lambda(C)$. It is clear that

$$\lambda(A)>\lambda(E)>\lambda(B) \quad \text{and} \quad \lambda(C)>\lambda(F)>\lambda(D) .$$

Therefore we have $y_{\mathrm{II}}^{+}(\lambda)=y_{\mathrm{II}}^{-}(\lambda)$ for some value of λ, and we denote this value of λ by λ_0.

In the case where $\lambda(E)=\lambda(F)=\lambda_0$, we have

$$(4.2)\qquad \begin{aligned} &|y_{\mathrm{II}}^{+}(\lambda)|<y_{\mathrm{I}}^{+}(\lambda) , \quad \text{for} \quad \lambda_0\leqq\lambda<a , \quad \text{and} \\ &|y_{\mathrm{II}}^{-}(\lambda)|<y_{\mathrm{I}}^{+}(\lambda) , \quad \text{for} \quad d<\lambda\leqq\lambda_0 , \end{aligned}$$

and

$$(4.3)\qquad \begin{aligned} &|y_{\mathrm{II}}^{+}(\lambda)|<|y_{\mathrm{I}}^{-}(\lambda)| , \quad \text{for} \quad b<\lambda\leqq\lambda_0 , \quad \text{and} \\ &|y_{\mathrm{II}}^{-}(\lambda)|<|y_{\mathrm{I}}^{-}(\lambda)| , \quad \text{for} \quad \lambda_0\leqq\lambda<c , \end{aligned}$$

where a, b, $\cdots$ denote the values of λ at A, B, $\cdots$.

We shall prove the inequalities (4.2) and (4.3) for the case in which $\lambda(E)$ is not equal to $\lambda(F)$. We may suppose, without loss of generality, that

$$\lambda(E)<\lambda(F) ,$$

which implies that $y_{\mathrm{II}}^{+}(\lambda_0)=y_{\mathrm{II}}^{-}(\lambda_0)>0$.

It is clear that inequalities (4.2) hold.

Now we shall prove (4.3) by contradiction. Suppose that there exists a λ such that $|y_{\mathrm{II}}^{-}(\lambda)|\geqq|y_{\mathrm{I}}^{-}(\lambda)|$ (see Fig. 7), and put

$$\lambda_1=\max\{\lambda;\ |y_{\mathrm{II}}^{-}(\lambda)|=|y_{\mathrm{I}}^{-}(\lambda)|,\ c>\lambda\geqq\lambda_0\}\ .$$

Let us denote by (x_1, η_1) and (x_2, η_2) the points of Γ corresponding to $y_{\mathrm{I}}^{-}(\lambda_1)$ and $y_{\mathrm{II}}^{-}(\lambda_1)$ respectively. We see that

$$G(x_1)=G(x_2) \quad \text{and} \quad \alpha<x_2<\beta_2<x_1<\beta_1\ .$$

Therefore $G(x_2)>G(\beta_2)>G(x_1)$ $(G(\beta_1)=G(\beta_2))$, which is impossible. This proves the second inequality of (4.3). We can prove, in the same way as above, the first inequality of (4.3).

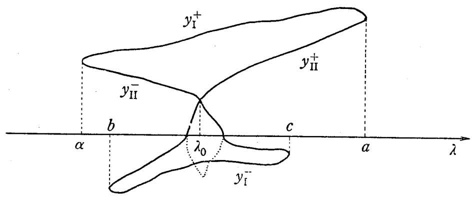

Fig. 7.

Now let T be the period of a closed path Γ of (3.4), i.e. $\Gamma=\{(x(t), y(t)): 0\leqq t\leqq T\}$. Then we have

$$\begin{aligned}
\int_0^T f(x(t))dt &= \Big(\int_a^b+\int_b^c+\int_c^d+\int_d^a\Big) f(x(t))\frac{dt}{d\lambda}d\lambda \\
&= -\Big\{\int_a^b\Big(\frac{1}{y_{\mathrm{II}}^{+}(\lambda)}\Big)^2 d\lambda+\int_b^c\Big(\frac{1}{y_{\mathrm{I}}^{-}(\lambda)}\Big)^2 d\lambda+\int_c^d\Big(\frac{1}{y_{\mathrm{II}}^{-}(\lambda)}\Big)^2 d\lambda+\int_d^a\Big(\frac{1}{y_{\mathrm{I}}^{+}(\lambda)}\Big)^2 d\lambda\Big\} \\
&= \Big(\int_b^{\lambda_0}+\int_{\lambda_0}^a\Big)(1/y_{\mathrm{II}}^{+})^2 d\lambda+\Big(\int_d^{\lambda_0}+\int_{\lambda_0}^c\Big)(1/y_{\mathrm{II}}^{-})^2 d\lambda \\
&\quad -\Big(\int_d^{\lambda_0}+\int_{\lambda_0}^a\Big)(1/y_{\mathrm{I}}^{+})^2 d\lambda-\Big(\int_b^{\lambda_0}+\int_{\lambda_0}^c\Big)(1/y_{\mathrm{I}}^{-})^2 d\lambda>0\ .
\end{aligned}$$

Here the last inequality follows from (4.2) and (4.3).

Therefore it follows from Poincaré's criterion for orbital stability that every closed path of (3.4) is orbitally asymtotically stable. However, two adjacent closed paths cannot be both orbitally asymptotically stable.

This completes the proof.

In concluding the paper, the author wishes to express his hearty thanks to Professor T. Kimura for his kind guidance and valuable advice.

References

[1] FURUYA, S.; On a nonlinear differential equation, *Comment. Math. Univ. Sancti Pauli*, **9** (1961), 97-113.

[2] SATÔ, Y.; On the existence of limit cycles of circuits containing one Esaki diode, *ibid.*, **18** (1970), 1-17.

[3] SANSONE, G. and CONTI, R.; *Non-linear differential equations*, Pergamon press, 1964, 334-341.

Department of Mathematics
Saitama University
Urawa, Japan

Errata

In Fig. 3, the point common to Γ and x-axis, lying on the right of E, should be denoted by C and that common to Γ and y-axis, below the origin, should be denoted by D.

In Fig. 5, separatrices Π- and Γ- should be depicted by the same heavy curve as the figure.

In Fig. 7, α should be read as d.